Measurement uncertainty relations

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Summary

Measurement uncertainty relations are quantitative bounds on the errors in an approximate joint measurement of two incompatible observables like position and momentum.

They can be seen as a generalization of the error/disturbance tradeoff first discussed heuristically by Heisenberg.

Such relations are closely connected with the more familiar preparation uncertainty relations, which constrain the sharpness of the distributions of the two observables in the same state.

My talk is based most directly on a paper with the same title (JMP **55** (2014) 042111 [29 pages]) together with **Paul Busch** (York) and **Reinhard Werner** (Hannover). Other related papers of BuLaWe:

- *Proof of Heisenberg's Error-Disturbance Relation*, PRL **111** (2013)160405 (2013) [5 pages].
- *Heisenberg uncertainty for qubit measurements*, PRA **89**, 012129 (2014)
- *Colloquium: Quantum root-mean-square error and measurement uncertainty relations*, RMP, in the press, arxiv:1312.4393v2 [quant-ph].

Heisenberg's original idea:

Heisenberg's 1927 intuitive ideas with a semiclassical analysis of the γ-ray thought experiment led him to the following conclusion:

A position measurement of an electron with an accuracy q_1 (resolution of the microscope) necessarily disturbs its momentum by an amount p_1 such that

$$p_1 q_1 \sim h. \tag{1}$$

Recent controversy

Some authors have recently claimed that Heisenberg proposed and even proved the following inequality

$$\textbf{Error}(A, \rho) \cdot \textbf{Disturbance}(B, \rho) \geq \frac{1}{2}|\langle [A, B] \rangle_\rho|, \tag{2}$$

with the notions of **Error**(A, ρ) and **Disturbance**(B, ρ) which turn this inequality wrong**!**

Such a claim is both absurd and historically wrong.

The proposed notions **Error**(A, ρ) and **Disturbance**(B, ρ) have only rather limited validity as a measurement error and disturbance caused by the measurement.

There can be **NO** error-disturbance relation of the form (2) **!**

PUR – Preparation uncertainty relations

– an example of the power of a theorem –

This controversy has nothing to do with the preparation uncertainty relations, like those proved originally by Kennard (1927), Weyl (or Pauli) (1928), and Robertson (1929):

$$\Delta_\rho(A) \cdot \Delta_\rho(B) \geq \frac{1}{2} |\langle [A, B] \rangle_\rho|, \tag{3}$$

Here Δ_ρ is the **standard deviation** of the measurement outcome distribution of a given observable in state ρ.

For $\Delta_\rho(A) \neq 0$, there is **NO TRUE VALUE** around which the measurement results are scattering.
- this is not a bug but a feature of quantum mechanics.

(3) is not an error-error or error-disturbance tradeoff relation.

To test a PUR like $\Delta_\rho(Q)\Delta_\rho(P) \geq \frac{1}{2}|\langle[Q,P]\rangle_\rho| = \frac{\hbar}{2}$

NO single object is subjected both to Q and P -**measurements**:

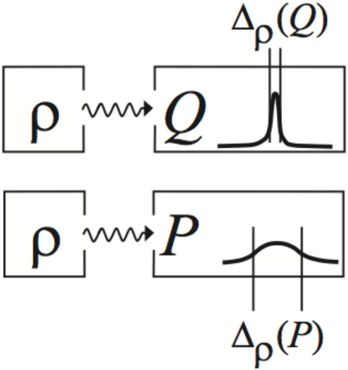


Figure: Scenario of preparation uncertainty

Recall also

- If

$$\Delta_\rho(A)\Delta_\rho(B) \geq c > 0 \quad \text{for all } \rho,$$

then A and B are (strongly) incompatible, they cannot be measured jointly or together (in any state).

- PUR gives no information how an A-measurement 'disturbs' another observable B.

- PUR gives no information on the possibility of measuring incompatible observables together approximatively.

…but it is a hint!

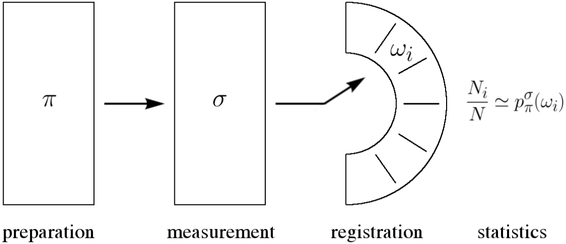
A study of measurement uncertainty relations and the claims of its alleged violation require careful mathematical and conceptual analysis:

– How to describe approximate measurements in quantum mechanics?

– How to quantify a measurement error and the disturbance caused by a measurement?

Operational basis of QM

– statistical causality –



- **states** as equivalence classes $[\pi]$ of preparations π,
- **observables** as equivalence classes $[\sigma]$ of measurements σ,
- **measurement outcome probabilities** $p_{[\pi]}^{[\sigma]}(X) \simeq n(X)/N$,
- the map $[\pi] \mapsto p_{[\pi]}^{[\sigma]}$ **preserves statistical mixing** of preparations π and thus of states $[\pi]$.

QM $=$ the Hilbert space quantum mechanics

- **Assume** that **states** $[\pi]$ are given as **density operators** ρ $[\rho \geq 0$, $\mathrm{tr}\,[\rho] = 1$, or the so-called pure states as unit vectors (wavefunctions)],

  then

$\Rightarrow$ **the sturcture of observables $[\sigma]$ is completely determined** [as positive operator measures $\mathsf{A} : X \mapsto \mathsf{A}(X)$, with outcome spaces $(\Omega, \mathcal{A})$, typically the real Borel spaces $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$];

$\Rightarrow$ **the form of the probability measures $p_{[\pi]}^{[\sigma]}$ is completely determined** [as the Born rule: $X \mapsto p_\rho^{\mathsf{A}}(X) = \mathrm{tr}\,[\rho\mathsf{A}(X)] = \mathsf{A}_\rho(X)$].

Fundamental features of quantum mechanics, 1

Single measurement results $\omega_i \in X$ have almost no meaning at all.
It is the whole measurement outcome statistics (distribution)

A_ρ [defined by the preparation and measurement]

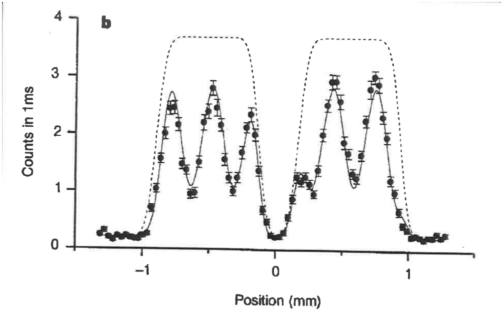
which is the result of a measurement.

Example: the statistics of a douple-slit experiment with single atoms

S.Dürr, T. Nonn, G. Rempe, *Nature* **395** (1998) 33-37.

A single spot on the screen (a measurement outcome) is of no special meaning.

The result of the measurement is the full position distribution Q_ρ on the screen [from which we can compute e.g. $\langle Q\rangle_\rho$ and $\Delta_\rho(Q)$].



Fundamental features of quantum mechanics, 2

A measurement changes, in general, the state of the system

$$\rho \mapsto \rho',$$

and thus also the measurement outcome distributions of observables

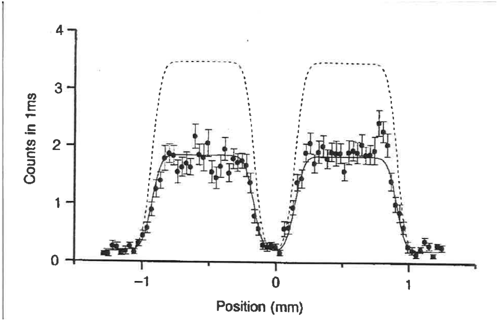
$$\mathsf{A}_\rho \mapsto \mathsf{A}_{\rho'} \equiv \mathsf{A}'_\rho.$$

This we see again in the statistics.

Douple-slit experiment with 'path marking'

- no need to read the mark-

The 'disturbance' felt by the atoms is seen in the statistics $(Q_\rho \mapsto Q_{\rho'})$.



By inspection: $\langle Q\rangle_\rho \simeq \langle Q\rangle_{\rho'}$ and $\Delta_\rho(Q) \simeq \Delta_{\rho'}(Q)$, though the distributions Q_ρ and $Q_{\rho'}$ differ hugely!

Joint measurements

A **bimeasurement** is any measurement (and thus observable M) which registers (with independent detectors) **pairs of outcomes**.

Let M_1 be the **partial observable** (marginal) obtained when the second outcome is ignored.

Let M_2 be the **partial observable** (marginal) obtained when the first outcome is ignored.

Observables (sharp or not) A and B can be measured **jointly** (together) if there is a bimeasurement M such that

$$\mathsf{A} = \mathsf{M}_1 \quad \text{and} \quad \mathsf{B} = \mathsf{M}_2.$$

Fundamental features of quantum mechanics, 3

There are pairs of observables that cannot be measured together.

Such pairs include the typical canonical pairs:

- position and momentum (in a given spatial direction),
- components of angular momentum,
- components of spin or polarization,
- number and phase,
- energy and time.

Error

In measuring an observable A with a measurement $\mathcal{M}$, the actually measured observable A' **may differ from** A.

An operational quantification $\Delta(\mathsf{A}', \mathsf{A})$ of the difference of the **approximator** A' from the **target** observable A is the **error** in measuring A with $\mathcal{M}$.

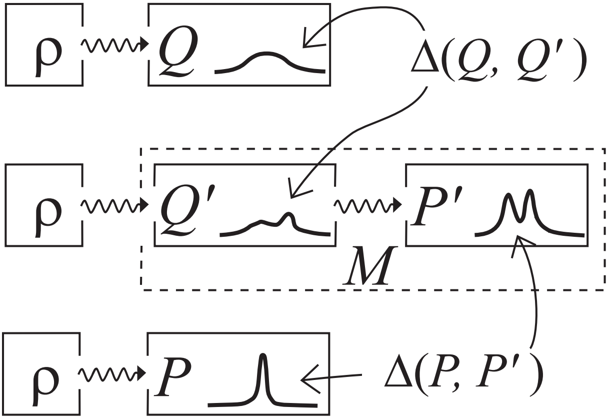
Disturbance

The measurement $\mathcal{M}$ causes a change (disturbance) in any other observable B, the disturbed observable B′ being uniquely determined from B by $\mathcal{M}$.

An operational quantification $\Delta(\mathrm{B}', \mathrm{B})$ of the difference of B′ from B is the **disturbance** of B caused by $\mathcal{M}$.

Heisenberg's scenario for $p_1 q_1 \sim h$.

The middle row shows an approximate position measurement Q' followed by a momentum measurement.



Approximate joint measurements

The error-disturbance scenario is just a special case of the approximate joint measurement scenario.

An **approximate joint measurement** of observables A and B is **ANY** measurement, with pairs of outcomes, and thus **any observable** M, **with two marginals** $\mathsf{M}_1, \mathsf{M}_2$, so that we may consider $\mathsf{M}_1/\mathsf{M}_2$ approximating A/B.

The errors $\Delta(\mathsf{M}_1, \mathsf{A})$ and $\Delta(\mathsf{M}_2, \mathsf{B})$ quantify the quality of M as an approximate joint measurement of A, B.

Measurement uncertainty relation for A, B is any inequality that excludes for any M the origin $\Delta(\mathsf{M}_1, \mathsf{A}) = 0 = \Delta(\mathsf{M}_2, \mathsf{B})$ and some region around it. For instance,

$$\Delta(\mathsf{M}_1, \mathsf{A}) \cdot \Delta(\mathsf{M}_2, \mathsf{B}) \geq c > 0, \quad \text{or} \;\; \Delta(\mathsf{M}_1, \mathsf{A})^2 + \Delta(\mathsf{M}_2, \mathsf{B})^2 \geq c' > 0.$$

How to define $\Delta(A', A)$?

In general, there is **no point in comparing individual measurement outcomes** of the approximator A' and the target observable A in each case.

But one may **compare the distributions** A'_ρ and A_ρ in all (or almost all) input states.

The number $\Delta(A', A)$ should thus quantify the difference between the distributions A'_ρ and A_ρ for all (or a relevant subset of) input states ρ.

- We need a **distance** $\Delta(\mathsf{A}'_\rho, \mathsf{A}_\rho)$ of the probability measures A'_ρ and A_ρ.
- Such a distance should behave correctly under ‘change of units’.
- This requires a corresponding distance in Ω.

  For the case $\Omega = \mathbb{R}$ this is

$$d(x, y) = |x - y|.$$

Monge-Kantorovich-Wasserstein distance (of order 2)

– minimizing the total cost of transportation –

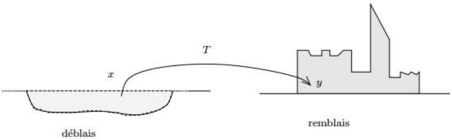


Fig. 3.1. Monge's problem of déblais and remblais

Kantorovich duality

– maximizing the win = minimizing the cost –

You may want a company to transport the extracted material to the consturction place:

Leonid Kantorovich's work from 1938 led to Nobel Prize for economics in 1975, jointly with Tjalling Koopmans, "for their contributions to the theory of optimum allocation of resources".

Optimal transport leads to a natural generalization of the Gaussian error **to compare any two probability measures**, like A'_ρ and A_ρ.

– for those who like to see it –

$$
\begin{aligned}
D^\gamma(\mathsf{A}'_\rho\, A_\rho) &= \left(\int |x-y|^2\, d\gamma(x,y)\right)^{\frac{1}{2}} \\
\Delta(\mathsf{A}'_\rho, \mathsf{A}_\rho) &=_{\text{cost}} \inf_{\gamma \text{ a coupling}} D^\gamma(\mathsf{A}'_\rho, \mathsf{A}_\rho) \\
&=_{\text{win}} \sup_{f,g} \left(\int g(y)\mathsf{A}'_\rho(dy) - \int f(x)\mathsf{A}_\rho(dx)\right)^{\frac{1}{2}}
\end{aligned}
$$

Compare with:

$$\Delta_\rho(A) = \Delta(A_\rho, \delta_{\langle A\rangle_\rho})$$

We take the **worst case** w.r.t ρ

$$\Delta(\mathsf{A}', \mathsf{A}) = \sup_{\rho} \Delta(\mathsf{A}'_{\rho}, \mathsf{A}_{\rho})$$

to represent **the error in measuring** A **with a measurement defining** A′.

This is a device figure of merit.

(Q, P): covariant phase space measurements are their approximate joint measurements.

They are measurements M (with two outcomes) which behave covariantly under phase space translations, that is, spatial translations and velocity boosts.

The partial measurements, the q- and p-marginals M_1 and M_2 are 'noisy' or 'unsharp' versions of position and momentum, smeared with the Fourier related densities Q_σ and P_σ (with σ defining M, that is, $M = M^\sigma$).

For them

$$\begin{aligned} \Delta(M_1, Q)\Delta(M_2, P) &= \left(\int q^2\, Q_\sigma(dq)\right)^{\frac{1}{2}}\left(\int p^2\, P_\sigma(dp)\right)^{\frac{1}{2}} \\ &\geq \Delta_\sigma(Q)\Delta_\sigma(P) \geq \frac{\hbar}{2}, \end{aligned}$$

with the lower bound attained with M^σ defined by the "oscillator ground state" $\sigma = |h_0\rangle\langle h_0|$.

General case

Theorem

Let M *be* **any** *approximate joint measurement of position and momentum such that the errors* $\Delta(\mathsf{M}_1, \mathsf{Q})$ *and* $\Delta(\mathsf{M}_2, \mathsf{P})$ *are finite. Then*

$$\Delta(\mathsf{M}_1, \mathsf{Q})\Delta(\mathsf{M}_2, \mathsf{P}) \geq \frac{1}{2}\,\hbar.$$

The lower bound is obtained by a covariant phase space measurement defined by the oscillator ground state.

Qubit observables

For a pair of ± 1-valued qubit observables A, B (associated with the directions $\mathbf{a}, \mathbf{b}$) we have:

Theorem

Let M *be* **any** *approximate joint measurement of the* ± 1*-valued qubit observables* A, B*. Then*

$$\Delta(\mathsf{M}_1, \mathsf{A})^2 + \Delta(\mathsf{M}_2, \mathsf{B})^2 \geq \sqrt{2}\left[||\mathbf{a} - \mathbf{b}|| + ||\mathbf{a} + \mathbf{b}|| - 2\right]. \tag{4}$$

This bound is tight and quantifies the degree of incompatibility of A, B*. It can be satisfied when the approximators* $\mathsf{M}_1, \mathsf{M}_2$ *are covariant.*

What goes wrong with the NO approaches

The experiments which claim the refutation of the Heisenberg uncertainty relations rely on the *noise operator* (NO) based notions of error and disturbance.

There are many equivalent expressions of this notion, e.g.,

$$\mathbf{Error}(A,\rho)^2 = \langle (A^{\mathrm{in}} - Z^{\mathrm{out}})^2 \rangle_{\rho\otimes\sigma} = \int (x-y)^2 \,\mathrm{Re}\,\mathrm{tr}\left[\rho \mathsf{A}'(dx)\mathsf{A}(dy)\right].$$

This immediately reveals that this notion **cannot** be determined from the actual measurement of the approximator A' and from the control measurement of the target A unless A' is compatible with A in which case it is an over estimation of $\Delta(\mathsf{A}'_\rho, \mathsf{A}_\rho)$.

A detailed analysis of the limitations of this notion as a measurement error is given in our Colloquium-paper.

Conclusion

Quantum mechanics allows one to

- define operationally significant notions of measurement error, for instance, those based on the Wasserstein deviation measures,
- prove for these notions Heisenberg-type of measurement uncertainty relations e.g. for position and momentum observables as well as for pairs of qubit observables.
- The method is general, but a generic relation is still lacking.
- The experiments which claim the violation of the Heisenberg measurement uncertainty relations rely on the noise operator based notions of error and disturbance. These are badly chosen definitions of measurement error, except when the quantities in question commute, in which case they over estimate the actual error.